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Third Order Convergence Iterative Method for Multiple Roots of Nonlinear Equation

Jamaludin, N. A. A.[1], Nik Long, N.M.A. *[1,2], Salimi, M.[3], and Ismail, F.[1,2]

[1] *Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, Malaysia*
[2] *Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*
[3] *Center for Dynamics and Institute for Analysis, Department of Mathematics, Technische UniversitÃt Dresden, Germany*

E-mail: nmasri@upm.edu.my
* *Corresponding author*

ABSTRACT

We present a new third order convergence iterative method for solving multiple roots of nonlinear equation, which requires one function evaluation and two evaluation of first derivative of function per step. Our present method free from second derivative function. Error term is proved to possess a third order method. Numerical experiments exhibit that our method gives the smallest error of bound per iteration and it is highly accurate as compared to other existing iterative methods.

Keywords: Multi-point iterative methods, Multiple roots, Nonlinear equations, Order of convergence, Root-finding.

1. Introduction

Solving nonlinear equations become the subject of interest to many researchers (see Chun et al. (2009), Dong (1982, 1987), Ferrara et al. (2015), Homeier (2009), Neta (2008), Victory Jr and Neta (1983), Zhou et al. (2013)). The well known modified Newton's method Schroeder and Stewart (1998) is written as

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}, \tag{1}$$

which is of second order convergence. Victory and Neta Victory Jr and Neta (1983) developed two-point third order iterative method for solving multiple roots which is given by

$$\begin{cases} y_k = & x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = & y_k - \dfrac{f(y_k)}{f'(x_k)} \cdot \dfrac{f(x_k) + Af(y_k)}{f(x_k) + Bf(y_k)}, \end{cases} \tag{2}$$

where $A = \mu^{2m} - \mu^{m+1}, B = -\dfrac{\mu^m(m-2)(m-1)+1}{(m-1)^2}$ and $\mu = \dfrac{m}{m-1}$. Equation (2) requires two evaluations of function, f and one evaluation of first derivative of function, $f^{'}$. Osada (1994) proposed the iterative method for finding the multiple roots of nonlinear equation, written as

$$x_{k+1} = x_k - \frac{1}{2}m(m+1)\frac{f(x_k)}{f'(x_k)} + \frac{1}{2}(m-1)^2\frac{f'(x_k)}{f''(x_k)}. \tag{3}$$

This iterative method requires one evaluation of function, f, two evaluations of f' and one evaluation of second derivative, f''. By combining Osada's method (3) and Euler-Chebyshev's method Traub (1977), Chun et al. Chun et al. (2009) produced the new iterative method for multiple zeros given by

$$x_{k+1} = x_k - \frac{m\big((2\gamma-1)m+3-2\gamma\big)}{2}\frac{f(x_k)}{f'(x_k)} + \frac{\gamma(m-1)^2}{2}\frac{f'(x_k)}{f''(x_k)} - \frac{m^2(1-\gamma)}{2}\frac{f(x_k)^2 f''(x_k)}{(f'(x_k))^3} \tag{4}$$

where,

$$A = \mu^{2m} - \mu^{m+1},\ B = -\frac{\mu^m(m-2)(m-1)+1}{(m-1)^2},\ \mu = \frac{m}{m-1},\ \gamma = -1 \text{ or } \frac{1}{2}.$$

Neta Neta (2008) developed the new iterative method for multiple zeros with the same number of function evaluations as in (2), written as

$$\begin{cases} y_k = & x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = & y_k - \dfrac{f(x_k)}{f'(x_k)}\left(\beta + \gamma\dfrac{f(y_k)}{f'(x_k)}\right), \end{cases} \tag{5}$$

where,

$\beta = m - \dfrac{m(m-\theta)}{\theta^2}$, $\gamma = \dfrac{m(m-\theta)}{\rho\theta^2}$, $\rho = \left(\dfrac{m-\theta}{m}\right)^m$ and $\theta \in \mathbb{R}$.

In this paper we develop the iterative method which is free from the evaluation of second derivative of functions.

2. Construction of Method

The well known Chebyshev's method for computing the simple iterative method is given by Petkovic et al. (2012)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f''(x_k)}{2f(x_k)}\left(\frac{f(x_k)}{f'(x_k)}\right)^2. \tag{6}$$

Equation (6) requires one evaluation of function, one evaluation of first derivative function and one evaluation of second derivative function. Let the Newton-type iterative method be

$$x_{k+1} = x_k - \frac{2m}{m+2}\frac{f(x_k)}{f'(x_k)}. \tag{7}$$

Expand $f'(x_{k+1})$ in (7) about $x = x_k$ yields,

$$f''(x_k) \cong \frac{(m+2)f'(x_k)\left[f'(x_k) - f'(y_k)\right]}{2mf(x_k)}. \tag{8}$$

Subtituting (8) into (6) gives

$$\begin{cases} y_k = & x_k - \dfrac{2m}{m+2}\dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = & y_k - \dfrac{(m+2)f(x_k)}{4m}\left[\dfrac{f'(x_k) - f'(y_k)}{(f'(x_k))^2}\right]. \end{cases} \tag{9}$$

We introduced the free disposable parameters, α and β, such that (9) becomes

$$\begin{cases} y_k = & x_k - \dfrac{2m}{m+2}\dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = & y_k - \dfrac{(m+2)f(x_k)}{4m}\left[\dfrac{\alpha f'(x_k) - \beta f'(y_k)}{(f'(x_k))^2}\right]. \end{cases} \tag{10}$$

3. Convergence Analysis

Theorem 1. Let $x^* \in D$ be a multiple roots of a sufficiently smooth function $f : D \subseteq R \to R$ defined on an open interval D with the multiplicity $m > 1$, which includes x_0 as an initial approximation of x^*. Then, the iterative methods defined by (10) has third order convergence when

$$\begin{cases} \alpha = \dfrac{m^3\left(m(m+4)+8\right)}{(m+2)^2}, \\ \beta = m^{-m+4}\,(m+2)^{m-1}, \end{cases} \tag{11}$$

with the error term

$$e_{n+1} = \frac{2c_1^2 e_n^3}{m^3} + O(e_n^4).$$

Proof. Let $e_n := x_n - x^*, e_{n,y} := y_n - x^*, c_i := \dfrac{m!}{(m+i)!}\dfrac{f^{m+i}(x^*)}{f^m(x^*)}, c_0 = 1, p = m+1, q = m+2, r = m-1$. Since $f(x^*) = 0$, Taylor expansion of f at x^* yields

$$f(x_n) = e_n^m\left(1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3\right) + O(e_n^4), \tag{12}$$

and

$$f'(x_n) = e^r\left(m + e_n p c_1 + e_n^2(q)c_2 + e_n^3\left((m+3)c_3 + O(e_n^4)\right)\right), \tag{13}$$

hence

$$e_{n,y} = y_n - x^* = \frac{me_n}{q} + \frac{2c_1e_n^2}{m(q)} + \frac{\left(-2(p)c_1^2 + 4mc_2\right)e_n^3}{m^2(q)} + O(e_n^4), \tag{14}$$

For $f(y_n)$ we have

$$f(y_n) = e_{n,y}^m\left(1 + c_1 e_{n,y} + c_2 e_{n,y}^2 + c_3 e_{n,y}^3\right) + O(e_{n,y}^4). \tag{15}$$

Substituting (12)-(15) into (10) gives the error term as

$$e_{n+1} = D_1 e_n + D_2 e_n^2 + D_3 e_n^3 + O(e_n^4), \tag{16}$$

where

$$D_1 = \frac{m}{m+2} + \frac{-m(m+2)\alpha + m^m(m+2)^{-m+2}\beta}{4m^3}, \tag{17}$$

$$D_2 = \frac{q^{-p}\left(m^2q^m\left(8m^2 + q^2\alpha\right) - m^m q^2\left(4 + mq\right)\beta\right)c_1}{4m^5}, \tag{18}$$

$$D_3 = \frac{1}{4m^7} q^{-p} \left(H + J m^m q^2 \beta \right), \tag{19}$$

$$H = -m^3 q^m \left(8m^2 + q^2 \alpha\right) \left(p c_1^2 - 2mc_2\right) \tag{20}$$

and

$$J = \left(\left(8 + mp(4 + mq)\right) c_1^2 - 2m^2(4 + mq)c_2 \right). \tag{21}$$

To obtain the convergence order of three, it is necessary to choose $D_i = 0$ $(i = 1, 2)$ which yield

$$\alpha = \frac{m^3 \left(m(m+4) + 8\right)}{(m+2)^2},$$

and

$$\beta = m^{-m+4} \left(m+2\right)^{m-1},$$

and the error terms become

$$e_{n+1} = \frac{2c_1^2 e_n^3}{m^3} + O(e_n^4),$$

which complete the proof.

4. Numerical Analysis

To analyse the performance of the proposed method, we conduct the numerical comparison by using different type of test functions listed in Table 1. We utilized the software package Mathematica 11 with 200 significant digit multi-precision and compute the error bound, (EB) the computational order of convergence (COC) see Weerakoon and Fernando (2000) and the approximated computational order of convergence (ACOC)as in Grau-Sánchez et al. (2010), which are defined respectively as $EB = |x_k - x^*|$,

$$\text{COC} \approx \frac{\ln |(x_{k+1} - \alpha)/(x_k - \alpha)|}{\ln |(x_k - \alpha)/(x_{k-1} - \alpha)|} \tag{22}$$

and

$$\text{ACOC} \approx \frac{\ln |(x_{k+1} - x_k)/(x_k - x_{k-1})|}{\ln |(x_k - x_{k-1})/(x_{k-1} - x_{k-2})|}. \tag{23}$$

The following are the list of existing third order iterative methods for multiple roots :

1. *Dong's method (DM)* Dong (1987), given by

$$\begin{cases} y_k &= x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \dfrac{f(x_k)}{\left(\dfrac{m}{m-1}\right)^{m+1} f'(y_k) + \dfrac{m-m^2-1}{(m-1)^2} f'(x_k)}. \end{cases} \tag{24}$$

2. *Chun et al.'s method (CBN)* Chun et al. (2009), given in (4) with the value of $\gamma = -1$.

3. *Hommier's method (HM)* Homeier (2009), given by

$$\begin{cases} y_k &= x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - m^2 \left(\dfrac{m}{m+1}\right)^{m-1} \dfrac{f(x_k)}{f'(x_k)} + m(m-1)\dfrac{f(x_k)}{f'(x_k)}. \end{cases} \tag{25}$$

4. *Zhou et al.'s method (ZCS)* Zhou et al. (2013), given by

$$\begin{cases} y_k = & x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = & x_k + m(m-2)\dfrac{f(x_k)}{f'(x_k)} - m(m-1)\left(\dfrac{m}{m-1}\right)^m \dfrac{f(y_k)}{f'(x_k)}. \end{cases} \tag{26}$$

5. *Ferrara et al.'s method (FSS)* Ferrara et al. (2015), given by

$$\begin{cases} y_k &= x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= x_k - \dfrac{\theta f(x_k)}{\theta f(x_k) - f(y_k)} \dfrac{f(x_k)}{f'(x_k)}, \end{cases} \tag{27}$$

   where $\theta = \left(\dfrac{-1+m}{m}\right)^{-1+m}$.

Table 2 shows that from test functions f_1-f_5 the absolute error of our PM method gives smallest value as compared to other. The value of COC and ACOC proved that our method posses as third order convergence method. Thus, our method is converge faster and give more accurate numerical results as compared to other existing iterative methods.

Table 1: List of test functions

Test functions f_n	Roots x^*	Multiplicity m
$f_1(x) = (\sin^2 x + x)^5$	0	5
$f_2(x) = (\ln(1+x^2) + e^{x^2-3x}\sin x)^6$	0	6
$f_3(x) = (x^3 + \ln(1+x))^7$	0	7
$f_4(x) = (x^6-8)^2 \ln(x^6-7)$	$\sqrt{2}$	3
$f_5(x) = (\ln(x^3-x+1) + 4\sin x - 1)^{10}$	1	10

Table 2: Error, COC and ACOC of test functions

Methods	PM	DM	CBN	HM	ZCS	FSS
$f_1, x_0 = 0.1$						
$\lvert x_1 - x^*\rvert$	$0.455e^{-3}$	$0.420e^{-3}$	$0.113e^{-2}$	$0.115e^{-2}$	$0.131e^{-2}$	$0.740e^{-3}$
$\lvert x_2 - x^*\rvert$	$0.379e^{-10}$	$0.314e^{-10}$	$0.215e^{-8}$	$0.242e^{-8}$	$0.424e^{-8}$	$0.364e^{-9}$
$\lvert x_3 - x^*\rvert$	$0.217e^{-31}$	$0.132e^{-31}$	$0.150e^{-25}$	$0.227e^{-25}$	$0.145e^{-24}$	$0.434e^{-28}$
COC	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC	3.0001	3.0000	2.9998	2.9997	2.9996	2.9999
$f_2, x_0 = 0.3$						
$\lvert x_1 - x^*\rvert$	$0.114e^{-1}$	$0.479e^{-1}$	$0.623e^{-1}$	$0.503e^{-1}$	$0.544e^{-1}$	$0.564e^{-1}$
$\lvert x_2 - x^*\rvert$	$0.205e^{-5}$	$0.116e^{-3}$	$0.665e^{-3}$	$0.478e^{-3}$	$0.711e^{-3}$	$0.178e^{-4}$
$\lvert x_3 - x^*\rvert$	$0.115e^{-16}$	$0.223e^{-11}$	$0.320e^{-9}$	$0.300e^{-9}$	$0.116e^{-8}$	$0.437e^{-14}$
COC	3.0000	3.0001	3.0011	3.0001	3.0002	3.0000
ACOC	3.0040	2.9490	3.2118	3.0735	3.0804	2.7447
$f_3, x_0 = 0.2$						
$\lvert x_1 - x^*\rvert$	$0.644e^{-3}$	$0.781e^{-2}$	$0.104e^{-1}$	$0.797e^{-2}$	$0.881e^{-2}$	$0.925e^{-2}$
$\lvert x_2 - x^*\rvert$	$0.189e^{-10}$	$0.376e^{-6}$	$0.967e^{-6}$	$0.283e^{-6}$	$0.0.418e^{-6}$	$0.702e^{-6}$
$\lvert x_3 - x^*\rvert$	$0.486e^{-33}$	$0.425e^{-19}$	$0.830e^{-18}$	0.134^{-19}	$0.0.482e^{-19}$	$0.316e^{-18}$
COC	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC	2.9995	2.9986	2.9934	2.9936	2.9921	2.9972
$f_4, x_0 = 1.5$						
$\lvert x_1 - x^*\rvert$	$0.956e^{-3}$	$0.221e^{-2}$	$0.422e^{-2}$	$0.299e^{-2}$	$0.329e^{-2}$	$0.329e^{-2}$
$\lvert x_2 - x^*\rvert$	$0.944e^{-8}$	$0.329e^{-6}$	$0.592e^{-5}$	$0.631e^{-6}$	$0.915e^{-6}$	$0.163e^{-5}$
$\lvert x_3 - x^*\rvert$	0.848^{-23}	$0.970e^{-18}$	$0.129e^{-13}$	$0.551e^{-17}$	$0.184e^{-16}$	$0.171e^{-15}$
COC	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC	3.0059	3.0134	3.0339	3.0088	3.0085	3.0207
$f_5, x_0 = 1.2$						
$\lvert x_1 - x^*\rvert$	$0.552e^{-4}$	$0.146e^{-2}$	$0.182e^{-2}$	$0.152e^{-2}$	$0.165e^{-2}$	$0.164e^{-2}$
$\lvert x_2 - x^*\rvert$	$0.337e^{-15}$	$0.688e^{-9}$	$0.171e^{-8}$	$0.832e^{-9}$	$0.0.116e^{-8}$	$0.109e^{-8}$
$\lvert x_3 - x^*\rvert$	$0.763e^{-49}$	$0.719e^{-28}$	$0.141e^{-26}$	$0.137e^{-27}$	$0.404e^{-27}$	$0.322e^{-27}$
COC	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
ACOC	3.0001	2.9999	2.9998	2.9998	2.9998	2.9999

5. Conclusion

In this paper, we developed a new third order convergence iterative method to obtain multiple roots for nonlinear equation. The new proposed method requires one function evaluation and two evaluation of the first derivatives. From the numerical, the new method shows rapid convergence compared to other existing methods in the scientific literature.

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